cogent appeal to personal initiative by providing a long list of mathematical papers that involve a minimum of technical treatment and presuppose, for the most part, only the topics usually given in undergraduate courses of study; and a shorter list of papers, involving more technical treatment, for the benefit of those who are ambitious to press on into somewhat higher ranges of mathematical reading. The object of the MONTHLY in selecting these papers is not simply to provide entertainment of a high character, nor even to be content with arousing *passive* interest in this intermediate field, but to *stimulate activity* on the part of college teachers—activity that may be reflected in the class room in the form of renewed enthusiasm, of incisive and critical presentation, of lively and keen interest that is always contagious among the students; and activity that may lead to *production*, to the contribution of similar papers for the benefit of others. If this supreme ambition on the part of the MONTHLY, which is meeting with most encouraging fulfillment in a narrow but gradually widening circle of influence, shall become in anywise general, then we believe that a fundamental and permanent contribution will have been made to the betterment of teaching in the colleges. To this end, every encouragement has been offered to all who will make even a small beginning. The department of Questions and Discussions provides an opportunity for minor contributions of varied character, and the department of Problems makes an appeal to large numbers. The really careful reviewing of a book is an activity that may easily lead to fruitful results. The writing of an acceptable article, historical, critical or technical, may be the beginning of a new epoch for the author; certainly persistence in such activity is capable of transforming a career, as attested by many notable examples.

2. **The Stimulation of Group Organization.** However much depends upon individual initiative and activity, it is certain that group activity is a kind of stimulus that seems to be necessary for the accomplishment of large things. There must be intercommunication and action based upon interchange of ideas. For this reason the MONTHLY has embraced every opportunity to encourage those organizations that have for their object the kind of development above described. For example, the reading club of the California teachers and the new organization among those secondary teachers of New Jersey who wish to discuss mathematical subjects at their meetings rather than mere pedagogical questions. But the aim of the MONTHLY is to see the college teachers of mathematics organized in every state, or even in some smaller groups, for the purpose of mutual assistance along all lines of interest in the collegiate field. A number of such groups are already under consideration, and one is being formally organized in the state of Kansas as this issue goes to press. The discussion at this meeting is on question 27 in the September MONTHLY, namely,

> "A certain college wishes to offer twelve hours of mathematics beyond the usual courses in analytic geometry and calculus. Considering only the needs of students intending to specialize in pure mathematics, what courses should make up the twelve hours offered?"

The MONTHLY congratulates these Kansas teachers and commends their example to all other such groups of college teachers throughout the country. With the

attention of college men fixed upon the varied and special problems of college work in mathematics, and with many group organizations for interchange of ideas, a new era will be inaugurated in this field.

3. **The Stimulation of National Organization.** The most natural climax of wide group organization is a national organization, such as was referred to in the October issue, and conversely, such a national organization will react on the formation of smaller groups and both will provide a far-reaching stimulus to individual activity. The MONTHLY would have welcomed the incorporation of such an organization within the American Mathematical Society, but since this is not to be, we look forward with high hopes and great enthusiasm to the organization of a new national society, and with more than four hundred charter members, we see no reason why commendable things should not be accomplished through this movement for the cause of mathematics in America.

HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

By FLORIAN CAJORI Colorado College.

X.

E. POST-CANTORION DISSENSIONS (Concluded).

With the advent of the new century, discussion on Zeno began to quiet down in France. We note only two articles. In 1907 O. Hamelin wrote on the "Arrow," but the interest of his article centers in what constitutes the most probable renderings of the Aristotelian text.[1] In 1909 a novel attempt to solve Zeno's puzzles was made by Dunan[2] in an article in which he retracts what he said on this subject in a pamphlet of 1884.[3]

He believes that the difficulties vanish, on the recognition that motion takes place through a space, one and indivisible, without succession and parts. He admits that such a proposition raises considerable difficulty, which cannot be removed except by long and elaborate metaphysics, of which he gives in his article only a bare sketch.

No less radical is the position of Henri Bergson. He holds that philosophy must get back to reality itself. Reality is supplied by intuition. Pure intuition, external or internal, is that of undivided continuity. Every movement, in as much as it is a passage from rest to rest, is in fact absolutely indivisible. Sight perceives the movement in the form of a line which is traversed, and this line, like all space, may be indefinitely divided. We must not confound the data of

[1] *L'année philosophique* de F. Pillon, Paris, 1907, pp. 39–44.

[2] "Zénon d'Élée et le Nativism" in *Annales de Philosophie Chrétienne*, 1909.

[3] *Les arguments de Zénon d'Élée contre le mouvement*, Nantes, 1884.

the senses, which perceive the movement as an undivided whole, with the artifice of the mind which divides into parts the path traversed. Says Bergson:[1]

"You substitute the path for the journey, and because the journey is subtended by the path you think that the two coincide. But how should a *progress* coincide with a *thing*, a movement with an immobility? . . . And from the fact that this line is divisible into parts and that it ends in points, we cannot conclude either that the corresponding duration is composed of separate parts or that it is limited by instants. The arguments of Zeno of Elea have no other origin than this illusion. They all consist in making time and movement coincide with the line which underlies them, in attributing to them the same subdivisions as to the line, in short in treating them like that line. In this confusion Zeno was encouraged by common sense, which usually carries over to the movement the properties of its trajectory, and also by language, which always translates movement and duration in terms of space. . . . But the philosopher who reasons upon the inner nature of movement is bound to restore to it the mobility which is its essence, and this is what Zeno omits to do. By the first argument (the Dichotomy) he supposes the moving body to be at rest, and then considers nothing but the stages, infinite in number, that are along the line to be traversed: we cannot imagine, he says, how the body could ever get through the interval between them. But in this way he merely proves that it is impossible to construct, *à priori*, movement with immobilities, a thing no man ever doubted. The sole question is whether, movement being posited as a fact, there is a sort of retrospective absurdity in assuming that an infinite number of points has been passed through. But at this we need not wonder, since movement is an undivided fact, or a series of undivided facts, whereas the trajectory is infinitely divisible. In the second argument (the Achilles) movement is indeed given, it is even attributed to two moving bodies, but, always by the same error, there is an assumption that their movement coincides with their path, and that we may divide it, like the path itself, in any way we please. Then, instead of recognizing that the tortoise has the pace of a tortoise and Achilles the pace of Achilles, so that after a certain number of these indivisible acts or bounds Achilles will have outrun the tortoise, the contention is that we may disarticulate as we will the movement of Achilles and, as we will also, the movement of the tortoise: thus reconstructing both in an arbitrary way, according to a law of our own which may be incompatible with the real conditions of mobility. The same fallacy appears, yet more evident, in the third argument (the Arrow) which consists in the conclusion that, because it is possible to distinguish points on the path of a moving body, we have the right to distinguish indivisible moments in the duration of its movement. But the most instructive of Zeno's arguments is perhaps the fourth (the Stadium) which has, we believe, been unjustly disdained, and of which the absurdity is more manifest only because the postulate masked in the three others is here frankly displayed. Without entering on a discussion which would here be out of place, we will content ourselves with observing that motion, as given to spontaneous perception, is a fact which is quite clear, and that the difficulties and contradictions pointed out by the Eleatic school concern far less the living movement itself than a dead and artificial reorganization of movement by the mind."

Bergson discusses the "Arrow" more fully in his *L'Évolution créatrive*, 1907, where he refers[2] to the absurdity of regarding movement as made up of immobilities. He says:

"Philosophy perceived this as soon as it opened its eyes. The arguments of Zeno of Elea although formulated with a different intention, have no other meaning. . . . Motionless in each point of its course, it is motionless during all the time of its moving. Yes, if we suppose that the arrow can ever *be* in a point of its course. Yes again, if the arrow, which is moving, ever coincides with a position, which is motionless. But the arrow never *is* in any point of its course. The most that we can say is that it might be there, in this sense, that it passes there and might stop there. . . . You fix a point C in the interval passed, and say that at a certain moment the arrow was at C. If it had been there it would have been stopped there, and you would no longer have had a flight from A to B, but *two* flights, one from A to C and the other from C to B, with an interval of rest. A single movement is entirely, by the hypothesis, a movement between two stops; if there are intermediate stops, it is no longer a single movement."

[1] H. Bergson, *Matter and Memory*, transl. by Nancy M. Paul and W. Scott Palmer, London, 1911, pp. 248, 250–253. The first French edition appeared in 1896.

[2] H. Bergson, *Creative Evolution*, transl. by A. Mitchell, London, 1911, pp. 325–327.

There have been many dissussions of Bergson. One writer endeavors to point out his errors by returning to the continuums of Artistotle and Thomas Aquinas.[1] Most pertinent to our topic are the criticisms by Bertrand Russell, of Cambridge, England, which are displayed by the following quotations:[2]

". . . it will be said, the arrow is where it is at any one moment, but at another moment it is somewhere else, and this is just what constitutes motion. Certain difficulties, it is true, arise out of the continuity of motion, if we insist upon assuming that motion is also discontinuous. These difficulties, thus obtained, have long been part of the stock-in-trade of philosophers. But if, with the mathematicians, we avoid the assumption that motion is also discontinuous, we shall not fall into the philosopher's difficulties. A cinematograph in which there are an infinite number of films, and in which there is never a *next* film because an infinite number come between any two, will perfectly represent a continuous motion. Wherein, then, lies the force of Zeno's argument? . . . Zeno assumes, tacitly, the essence of the Bergsonian theory of change. That is to say, he assumes that when a thing is in process of continuous change, even if it is only change of position, there must be in the thing some internal *state* of change. The thing must, at each instant, be intrinsically different from what it would be if it were not changing. He then points out that at each instant the arrow simply is where it is, just as it would be if it were at rest. Hence he concludes that there can be no such thing as a *state* of motion, and therefore, adhering to the view that a state of motion is essential to motion, he infers that there can be no motion and that the arrow is always at rest. Zeno's argument, therefore, though it does not touch the mathematical account of change, does, *prima facie*, refute a view of change which is not unlike M. Bergson's. How, then, does M. Bergson meet Zeno's argument? He meets it by denying that the arrow is ever anywhere. After stating Zeno's argument, he replies: 'Yes, if we suppose that the arrow can ever *be* in a point of its course. Yes again, if the arrow, which is moving, ever coincides with a position, which is motionless. But the arrow never *is* in any point of its course.' (C. E., p. 325.) This reply to Zeno, or a closely similar one concerning Achilles and the Tortoise, occurs in all his three books. Bergson's view plainly, is paradoxical; whether it be *possible*, is a question which demands a discussion of his view of duration. His only argument in its favor is the statement that the mathematical view of change 'implies the absurd proposition that movement is made of immobilities.' (C. E., p. 325.) But the apparent absurdity of this view is merely due to the verbal form in which he has stated it, and vanishes as soon as we realize that motion implies relations. A friendship, for example, is made out of people who are friends, but not out of friendships. . . . So a motion is made out of what is moving, but not out of motions. It expresses the fact that a thing may be in different places at different times, and that the places may still be different however near together the times may be. Bergson's argument against the mathematical view of motion, therefore, reduces itself, in the last analysis, to a mere play upon words." . . .

"Mathematics conceives change, even continuous change, as constituted by a series of states; Bergson, on the contrary, contends that no series of states can represent what is continuous, and that in change a thing is never in any state at all." . . .

"One of the bad effects of an anti-intellectual philosophy, such as that of Bergson, is that it thrives upon the errors and confusions of the intellect. Hence it is led to prefer bad thinking to good, to declare every momentary difficulty insoluble, and to regard every foolish mistake as revealing the bankruptcy of intellect and the triumph of intuition. . . . As regards mathematics, he has deliberately preferred traditional errors in interpretation to the more modern views which have prevailed among mathematicians for the last half century."

Thus it is seen that among recent French philosophers the Cantor continuum has been neglected and no satisfactory substitute has been advanced.

A treatment of the "Achilles" altogether different from that hitherto given by either philosophers or mathematicians is given by Russell.[3] After explaining infinite number and the modern continuum, he says in the *International Monthly:*

[1] T. J. Gerrard, *Bergson, an Exposition and Criticism*, London and Edinburgh, 1913, pp. 23 ff.

[2] B. Russell, "The Philosophy of Bergson," *The Monist*, July, 1912. Russell's references (C. E.) are to the English translations of Bergson's *Creative Evolution*.

[3] See B. Russell, *Principles of Mathematics*, 1902; "Recent work on the Principles of Mathematics" in the *International Monthly*, Vol. IV, 1901, pp. 83–101.

"We can now understand why Zeno believed that Achilles cannot overtake the tortoise and why as a matter of fact he can overtake it. We shall see that all the people who disagreed with Zeno had no right to do so, because they all accepted premises from which his conclusion followed. . . . Then he [Achilles] will never reach the tortoise. For at every moment the tortoise is somewhere, and Achilles is somewhere; and neither is ever twice in the same place while the race is going on. Thus the tortoise goes to just as many places as Achilles does, because each is in one place at one moment, and in another place in another moment, and in another at any other moment. But if Achilles were to catch up with the tortoise, the places where the tortoise should have been, would be only part of the places, where Achilles would have been. Here, we must suppose, Zeno appealed to the maxim that the whole has more terms than the part. Thus if Achilles were to overtake the tortoise, he would have been in more places than the tortoise; but we saw that he must, in any period, be in exactly as many places as the tortoise. Hence we infer that he can never catch the tortoise. This argument is strictly correct, if we allow the axiom that the whole has more terms than the part. As the conclusion is absurd, the axiom must be rejected, and then all goes well. But there is no good word to be said for the philosophers of the past two thousand years and more, who have all allowed the axiom and denied the conclusion. The retention of this axiom leads to absolute contradictions, while its rejection leads only to oddities."

The conjectures which Russell makes on the history of the "Achilles" are, in the main, without foundation. There is no historical evidence for believing that Zeno based the "Achilles" on the doctrine that the whole is greater than any of its parts. Aristotle bases Zeno's argument on the assertion that a line or distance cannot be reduced by any process of successive division to elements that are mathematical points. Russell's version of the paradox is what Zeno might have said, but did not actually say. It is far simpler to explain than is that of Zeno. Assent can be readily secured to the fact that, in infinite aggregates, the whole is not greater than certain of its parts.[1]

That Russell's argument, though correct in itself, does not meet the exact difficulty experienced by many persons, is brought out by C. D. Broad,[2] who points out that the "Achilles" rests on the false assumption that "what is beyond every one of an infinite series of points, must be infinitely beyond the first point of the series." Broad considers it important even at this time to settle this controversy, "because it and Zeno's other paradoxes have become the happy hunting-ground of Bergsonians and like contemners of the human intellect." What makes infinite divisibility a stumbling block to so many is the fact that appeal is made to sensory intuition and imagination—the very faculty of the mind which proves itself unable to cope with the problem. But our powers of analysis penetrate realms of thought beyond the reach of the imagination, and it is in that territory that the arguments of Zeno are made to surrender their mysteries.

B. Russell took great interest also in the "Arrow." In the *International Review* he remarked:

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[1] In this connection a story told by De Morgan may be of interest. He relates "a tradition of a Cambridge professor who was once asked in a mathematical discussion, 'I suppose you will admit that the whole is greater than its part?' and who answered, "not I, until I see what use you are going to make of it.'" The danger of unintended implications is illustrated by an author who remarked that Gibbon always had a copy of Horace in his pocket and often in his hand, from which it would seem to follow that Gibbon's hand was sometimes in his pocket.

[2] *Mind*, Vol. 38, 1913, pp. 318, 319.

"Weierstrass, by strictly banishing from mathematics the use of infinitesimals, has at last shown that we live in an unchanging world, and that the arrow in its flight is truly at rest. Zeno's only error lay in inferring (if he did infer) that, because there is no change, therefore the world is in the same state at any one time as at any other. . . . Weierstrass has been able, by embodying his views in mathematics, where familiarity with truth eliminates the vulgar prejudices of common sense, to invest Zeno's paradoxes with the respectable air of platitudes." Elsewhere Russell expresses much the same idea by the statement that "a variable does not vary."[1]

That a variable cannot reach its limit is still widely held. In 1907 R. B. Haldane presented this as the teaching of mathematics, in a presidential address to the Aristotelian Society, entitled "The Methods of Modern Logic and the Conception of Infinity." In a review of this address, B. Russell says[2] that this property "belongs to limits of a certain particular sort," which constitute "an extremely special case, not realized in most of the series in which limits exist."

The creation of the theory of sets and of the Cantor continuum lead to modified definitions of the limit. In this theory the concept of a limit was divorced from the idea of quantity and measurement. The question whether the variable reaches its limit or not is ignored as being of no interest. Whether it reaches its limit or not depends upon the nature of the variation in a particular case; the sequence of values may include the limit, or it may not. The *limiting point* of a set of points is one for which every interval, however small, containing the limiting point, encloses a point of the set, distinct from the limiting point itself. A *limit* is merely the arithmetical equivalent of the limiting point in geometry. The introduction of a transfinite number as a limit has carried with it still further modification of the idea of a limit. Small intervals do not fit here. Says Bertrand Russell: "If we consider the whole series of integers, finite and infinite, arranged in order of magnitude, then the class of finite integers, considered as part of this series, has an upper limit, namely the smallest of the infinite integers (which is the number of finite integers)." Here there is no "negligible difference" between variable and limit; "the difference between the finite integers and their limit remains constant and infinite." Again he says: "A limit must not be conceived as something to which the successive terms of the class approach indefinitely near; they may all be at an infinite distance from the limit, or at a distance which remains permanently greater than some given finite distance; or the series concerned may be one in which there is no such thing as distance or difference." His definition of a limit is as follows: "Given any series, and a class α of terms belonging to the series, a term x belonging to the series is called the *upper limit* of α if every term of α precedes x, and every term of the series which precedes x precedes some member of α." He gives a similar definition for *lower limit*.[3] It is to be observed that the modern definitions of a limit are free from the concept of the old-time infinitesimal.

As now we pause and look backward, we see that a full and logically correct explanation of Zeno's arguments on motion has been given by the philosophers of mathematics. Looking about us, we see that the question is still regarded as

[1] B. Russell, "Mr. Haldane on Infinity," *Mind*, Vol. 33, London, 1908, p. 240.
[2] *Mind*, Vol. 33, 1908, p. 239.
[3] B. Russell, "Mr. Haldane on Infinity," *Mind*, Vol. 33, London, 1908, pp. 240, 241.

being in an unsettled condition. Philosophers whose intellectual interests are remote from mathematics are taking little interest in the linear continuum as created by the school of Georg Cantor. Nor do they offer a satisfactory substitute. The main difficulty is not primarily one of logic; it is one of postulates or assumptions. What assumptions are reasonable and useful? On this point there is disagreement. Cantor and his followers are willing to assume a continuum which transcends sensuous intuition. Others are not willing to do so. Hence the divergence. In the Koran there is a story that, after the creation of Adam, the angels were commanded to make him due reverence. But the chief of the angels refused, saying: "Far be it from me a pure spirit to worship a creature of clay." For this refusal he was shut out from Paradise. The doom of that chief, so far as the mathematical paradise is concerned, awaits those who refuse to examine with proper care the massive creation by our great mathematicians, without which the tiniest quiver of a leaf on a tree remains incomprehensible.[1]

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MATHEMATICAL MEETINGS IN CALIFORNIA.

I. THE TWENTY-SECOND SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The American Mathematical Society met for its twenty-second summer meeting as announced by the Society, on August 3, 1915, at the University of California in Berkeley. The first meeting was in conjunction with Section A of the American Association for the Advancement of Science, on Tuesday morning. Professor Keyser, of Columbia University, delivered an address on the human significance of mathematics, and Director Hale, of the Mount Wilson Solar Observatory, delivered an address on the work of a modern observatory. The attendance was very large and included members of the American Mathematical

[1] Since the completion of this article there has appeared Bertrand Russell's *Our Knowledge of the External World as a Field for Scientific Method in Philosophy*, Open Court Company, 1914, in which much attention is given to Zeno's arguments. An article on Zeno by Philip E. B. Jourdain will soon appear in *Mind*.

Society and the American Astronomical Society as well as members of the Association at large. At least 250 persons were present. Following this session the two societies enjoyed a luncheon at the University Faculty Club as the guests of Professors Leuschner, Haskell and E. B. Lewis.

The first separate session of the Society was held Tuesday afternoon at the University of California, where papers were presented by Professor L. E. DICKSON, University of Chicago; Professor C. J. DE LA VALLÉE POUSSIN, University of Louvain; Mr. A. R. SCHWEITZER, Chicago, Ill.; Dr. NATHAN ALTSHILLER, University of Colorado; Professor L. J. RICHARDSON, University of California; Dr. DUNHAM JACKSON, Harvard University; Dr. W. W. KÜSTERMANN, University of Michigan; Professor G. A. MILLER, University of Illinois; Professor H. S. WHITE, Vassar College; and Professor M. W. HASKELL, University of California.

The members present were largely from the Pacific coast, although a few from the eastern states were present. The lists of those present will appear as usual in the BULLETIN of the American Mathematical Society.

On Wednesday the Society met at Stanford University and papers were presented by Dr. B. A. BERNSTEIN, University of California; Dr. C. A. FISCHER, Columbia University; Mr. A. R. WILLIAMS, and Professor L. M. HOSKINS, Stanford University. Since this completed the program the other sessions announced were not necessary.

Returning to Berkeley Wednesday evening the society enjoyed dinner again in conjunction with the American Astronomical Society at the Hotel Oakland in Oakland.

On Friday about 50 members of the two societies took a very delightful trip to the Lick Observatory on Mount Hamilton, which was reached by automobiles from San José, and after remaining overnight in San José, enjoyed the hospitality of Mrs. Hearst on the return trip to San Francisco.

A more detailed account of this meeting will of course appear in the BULLETIN of the American Mathematical Society. To those easterners who were present the occasion was particularly delightful and was a revelation of the possibilities of the Coast both as a mathematical center and as an enjoyable meeting place.

II. OTHER MEETINGS AT CALIFORNIA.

In addition to the meeting of the American Mathematical Society in California, a large number of other meetings were held in connection with the exposition, the meetings themselves being largely in Berkeley, the seat of the University of California, or in Oakland, which immediately adjoins Berkeley.

Among the meetings of particular interest to the readers of the MONTHLY would be the American Association for the Advancement of Science, which was held during the entire week of August 1–7, and which will be reported in detail in the *Proceedings* of the Association. The attendance at this meeting was very large, and the members of the various component societies thought that the occasion was well chosen for a meeting. This is the more remarkable on account of the fact that the meeting was specially called, since the Association now has its regular meetings during the Christmas holidays.

Another gathering of peculiar interest was the meeting of the Association of American Agricultural Colleges and Experiment Stations, August 11–13, at Berkeley. The name of this association may seem to indicate that it is interesting only to agriculturalists but attention should be called to the fact that the Association includes all of the so-called Land-Grant colleges, which receive aid under various national acts from the United States government. Thus many state universities were represented at this meeting, and such institutions as the Massachusetts Institute of Technology. A subsidiary association of considerable importance to those interested in mathematics was the Land Grant College Engineering Association, which met at the same time. To those who are not aware of the purposes of these two associations it may be pointed out that one very important project before them at present is to secure government aid for the establishment of engineering experiment stations in the various land-grant colleges.

Finally, the meeting of the National Education Association held in Oakland, August 16–28, will be of interest to those who have to do with secondary education either directly or indirectly. No full account of this meeting is possible and we must content ourselves with the statement that it was a very large meeting and that a full report of it will be published by the Association itself. One discussion that occurred there which will be of interest to all who read the MONTHLY was concerned with the establishment of six-year high schools—a movement which would seem to be well under way and to promise great changes and considerable success in the near future.

A large number of other meetings of educational importance were held, including for example the Association of American Universities and the Association of American State Universities.

On the whole the meetings held this summer in California will certainly take rank as one of the most important groups of meetings which have ever been held in the United States.

HISTORY OF MATHEMATICS.

By G. A. MILLER, University of Illinois.

In February, 1640, Descartes wrote as follows: "I am accustomed to distinguish two things in the mathematics, the history and the science. *By the history I mean whatever is already discovered, and is committed to books.* And by the science, the skill of resolving all questions, and thence by investigating by our own industry whatever may be discovered in that science by human ingenuity. He who possesses this faculty has but little need of other assistance, and may therefore be properly called self-sufficient. Now it is much to be wished that this mathematical history, which lies scattered through many volumes, and is not yet entire and complete, were to be all collected into one book."[1]

[1] *The Philosophical Transactions of the Royal Society of London* (Abridged), Vol. 2, 1809, p. 533.

Descartes's definition of history of mathematics, implied by the sentence of the preceding quotation which was italicized by the present writer, may at first surprise many of those who have acquired their idea of the meaning of the term history of mathematics from the modern books bearing this title. A second thought will doubtless reveal difficulties in the way of defining clearly what we really mean by this term. Are the mathematical developments of the first decade and a half of the present century a part of the history of mathematics? If this is the case, should we not begin the study of the history of mathematics by the study of this period?

To direct attention to the feasibility of beginning a study of the history of mathematics near its border which is closest to us, it may be interesting to recall a few mathematical activities which were inaugurated since the beginning of the present century. The one which is probably of most interest to the average teacher is the work done under the general direction of the *International Commission on the Teaching of Mathematics*, which was created during the fourth international congress held at Rome in April, 1908. At first it was intended that this commission should confine its work to secondary mathematics, but it soon appeared desirable to include all mathematical instruction in the scope of its investigators.[1] The central committee is now composed of seven men representing seven leading countries as follows: F. Klein, Germany; G. Greenhill, England; D. E. Smith, United States; H. Fehr, Switzerland; G. Castelnuovo, Italy; E. Czuber, Austria; and J. Hadamard, France.

The magnitude of the work accomplished during the first six years of the existence of this Commission is partly exhibited by the fact that over ten thousand pages of reports prepared under its general direction were issued during this period, according to a statement made at the meeting held at Paris in April, 1914. These reports relate mainly to the materials and methods of teaching mathematics in sixteen different countries, including all those which lead in scientific activities, and they constitute a most valuable addition to the literature on the development of mathematics in various modern countries. Several additional reports have appeared since the given meeting, and it seems probable that the work inaugurated by this Commission will bear fruit for a long time to come.

Another large mathematical undertaking inaugurated during the first decade and a half of the present century is the publication of the great mathematical encyclopedia, entitled *Encyclopédie des Sciences Mathématiques*, the first part of which was published in 1904. More than thirty large volumes of this great encyclopedia have been planned and over five thousand pages have already been published. Unfortunately the great European war has greatly delayed this publication, which promises to become the largest and most valuable mathematical encyclopedia that has ever been written.

A considerable number of mathematical periodicals have been started during the period under consideration. About the beginning of this period the *Trans-*

[1] J. W. A. Young gave an account of the work of this Commission during its first four years, in this MONTHLY, Vol. 19 (1912), p. 161.

actions of the American Mathematical Society was started and it has enjoyed the support of leading American mathematicians from the start. The high standards and careful editorship of this periodical have been of the greatest value in the development of research activity of a high order in our midst. Quite recently an important general scientific journal, devoting considerable attention to mathematics, was started in our country, viz., the *Proceedings of the National Academy of Sciences*. The first number appeared in January of the present year, and the periodical at once received the support of leading investigators in the various scientific fields.

In May, 1911, there was started a new Spanish mathematical journal, entitled *Revista de la Sociedad Matematica Española*, which is of special interest to Americans in view of the fact that the Spanish language is used in large parts of our continent. This periodical is the official organ of the national mathematical society of Spain, which was organized about a month before the periodical was started. It is to be hoped that this society and its journal will do much to organize the mathematical work among the people using the Spanish language, especially since these people have not taken an active part in the development of mathematics during recent centuries, having remained far behind the Italians in this regard.

Another important mathematical periodical entitled *Tôhoku Mathematical Journal* was started in 1911 at Sendai, Japan. It invited from the beginning contributions in English, French, German, Italian, and Japanese, but most of its articles thus far have been in English. This is the first journal devoted mainly to modern advanced mathematics which has been published in Japan, and its international character should do much to advance the interests of higher mathematics in that country.

In view of the fact that Asia took practically no part in the development of mathematics in modern times up to the beginning of the twentieth century, it is of interest to note another thriving mathematical journal started in Asia during the period under consideration. This periodical is entitled *The Journal of the Indian Mathematical Society*, and was started at Madras, India, in February, 1909. The fact that it is the official organ of a society founded in 1907 for the advancement of mathematical study and research in India makes it the more interesting and increases its opportunities for usefulness. Another Indian mathematical periodical was started in 1909, under the title *Bulletin of the Calcutta Mathematical Society*, but only four numbers of this journal, which was to be a quarterly, have been issued thus far.

One of the most noteworthy features of the period under consideration is the rapidly increasing interest in questions relating to the teaching of mathematics, and the great success of the International Commission on the Teaching of Mathematics was doubtless largely due to the fact that the time was ripe for vigorous advances along this line. Reforms of various kinds and of far-reaching significance have received widespread attention. A considerable number of new journals devoted mainly to methods of teaching were established during the

decade and a half under consideration. In our own country *School Science and Mathematics* and *The Mathematics Teacher* may serve as illustrations. The former of these is a continuation of *School Science*, the first number of which appeared in March, 1901; while the latter began to appear as a quarterly in September, 1908.

Perhaps no other undertaking started during the period under consideration exhibits the spirit of this period so clearly as the commencement of the publication of the collected works of the great Swiss mathematician, Leonard Euler. Various efforts had been made earlier to publish these very extensive works but these efforts failed on account of the large amount of money required for the publication. In September, 1909, the Swiss Society of Naturalists, having received through national and international subscriptions and through donations about one hundred thousand dollars for this purpose, announced that it would undertake this great publication. Two years later the first volume was published, and several other volumes followed in rapid succession, but it soon appeared that, in view of the discovery of new MSS., the entire publication would cost nearly twice as much as the original estimate, and that it would fill more than forty-five large volumes. This led to the formation of a unique international mathematical society, called the *Leonard Euler-Gesellschaft*, whose main object is to aid this publication.

The developments which have been noted are of a general nature and they constitute merely evidences of the fact that there was real mathematical growth during the period under consideration. It would be of more interest to consider some of the mathematical advances themselves, but these are too numerous and too extensive to be described in a brief article. If it is observed that the *Jahrbuch über die Fortschritte der Mathematik* fills annually a volume of over a thousand pages in giving titles and brief reviews of the new literature on mathematics, it results that for the decade and a half under consideration it would require more than fifteen thousand pages to present even such a limited consideration of the mathematical progress as is contained in the given review. Hence it is clear that a history of the development of mathematics during the first decade and a half of the twentieth century might well fill many volumes.

The main object of the present article is to raise the question whether such books on the history of mathematics as Ball, Cajori, Cantor, etc., are not apt to convey a very incorrect notion of what the history of mathematics really is, and of what developments should be embodied in a first course on this subject. In fact, Cantor's *Vorlesungen über Geschichte der Mathematik* are confined to the developments which preceded the nineteenth century, and hence they do not touch the period in which most of our present mathematical literature originated. Possibly some of the other books could be used to the best advantage as text-books by beginning near the end and working forward, but a really inspiring course in the history of mathematics would probably have to be based largely upon the literature contained in the recent journals.

A course in the history of mathematics should not tend to create a veneration

of the past at the expense of an appreciation of the present. It would be better to be ignorant of the Pythagorean school than to be ignorant of the modern mathematical schools. It would be better to be ignorant of the Greek Eleatic school than to be ignorant of some of the fundamental results in the modern theory of aggregates. It would be better to be able to name with some intelligence ten of the most eminent living mathematicians than to be able to name that number of ancient Arabians who helped to preserve and to transmit the mathematical lore of the ancient Greeks.

One difficulty about beginning a course in the history of mathematics with the developments of the last ten or fifteen years is that the new theorems and theories have not yet established their true value in the permanent literature. Mathematical fashions are changeable, but mathematical worth is permanent. Recent mathematical history is apt to be affected by the fashions while the older mathematical history is based upon established permanent values. These objections do not apply so strongly to a study of the various present activities which were inaugurated with a view to furthering mathematical interests, and these activities should be well understood by all those who teach mathematics.

It is also true that it is somewhat more difficult to keep in touch with the recent developments than it is to study once for all some accounts of the older developments. This difficulty has been greatly reduced in recent years by the increase in the periodical literature and by the various aids to make rapid surveys of the main advances in various fields. There is less and less excuse for living entirely in the mathematical past. The growing dynamic elements of mathematics naturally appeal to a large number, especially to the younger people, and these elements are best understood if they are observed in their natural surroundings and in real life.

The given quotation from Descartes seems to imply that he thought that all the mathematical history of his day could have been collected into *one* book. We noted above that the large French mathematical encyclopedia is expected to fill more than thirty volumes, and this will not include all the known mathematical results. If it is observed that more than two thousand original mathematical articles are published every year, it is clear that mathematical history is growing more rapidly than one man could write it. Hence such a thing as a complete mathematical history seems out of question. It is, however desirable to know something about this history, and especially about that part which lies closest to us.

It is of interest to inquire what distinguishes the history of mathematics from other mathematical writings, in case we assume that there is a difference, as is commonly done. According to Descartes's view practically all the literature which appears in our better journals of mathematics should be regarded as historical, while many others would probably be inclined to contend that only a small part of this literature should be classed with history. All might agree on the statement that every mathematical advance is making mathematical history, but some would probably hold that this advance would become history through

some process of maturing, while the process itself might be vague and hence undefinable. Judging from the writings which are now commonly classed under mathematical history it would appear that the chronological element was generally considered an essential element in a direct historical paper. In an indirect historical paper,—for instance, one discussing how mathematical history should be written—this element would not need to be present.

Next in importance to the chronological element in the usual historical presentation of a mathematical subject is the human element. Even the mere names of those who have enriched mathematical thought by pointing out logical steps leading to views of unusual beauty or to regions of unusual fruitfulness serve to establish a sense of comradeship. This sense is intensified by more or less complete biographical notices. It should, however, be observed that these names and these biographical sketches serve other useful purposes. The names of great mathematicians may be used to unify varied mathematical results, while the biographical sketches serve to unify mathematical knowledge and knowledge relating to other lines of thought.

While these considerations may throw some light on the term history of mathematics they are not intended to convey the idea that this term could be defined in a perfectly satisfactory manner. Like some other useful mathematical terms, the term history of mathematics will probably always remain without a real definition. While one may not know where mathematical history begins, yet there are some writings which all agree to classify under history and others which few would classify under this heading. On the other hand, there seems to be a strong tendency towards increasing the historical element in modern mathematical writings so that the writings which are zero per cent. history are becoming less and less common. It is to be hoped that the mathematical teaching which devotes zero per cent. of the time to history, and the mathematical history which includes zero per cent. of the present-day mathematical activities will also become less and less common.

ON THE CIRCLES OF APOLLONIUS.

(Concluded)

By NATHAN ALTSHILLER, University of Colorado.

(By some unaccountable oversight this last page of DR. ALTSHILLER'S article was omitted from the October issue. EDITORS.)

12. The three pairs of points A, A'; B, B'; C, C' of ω are perspective from K; they belong therefore to the same involution on ω, K and l being respectively the pole and the axis of the involution[1] (9A). Hence the couples of lines AB, $A'B'$; BC, $B'C'$; AC, $A'C'$ meet on l. Now, C_3, C_1, C_2 are the points common to l and the lines AB, BC, AC respectively. Hence:

[1] Russell, *loc. cit.*, pp. 217, 218.

The lines $A'B'$, $B'C'$, $A'C'$ meet the lines AB, BC, AC respectively in the centers of the Apollonian Circles.

13. The points A, B, A', B' are the vertices of a complete quadrangle inscribed in ω. The polars s_3 and l, with respect to ω (8, 9), of the diagonal points $C_3 \equiv (AB, A'B')$ (12) and $K \equiv (AA', BB')$ (7) intersect in the third diagonal point[1] $(AB', A'B)$, *i. e.*, this point is identical with P_3 (11). Similarly for the poles P_1 and P_2. Hence:

The pairs of lines AB', $A'B$; BC' $B'C$; AC', $A'C$ meet respectively in the poles P_3, P_1, P_2 of the Brocard diameter with regard to the Apollonian circles.

14. The triangles KP_1C_1, KP_2C_2, KP_3C_3, are self-conjugate with respect to ω (13), and therefore:

The center of an Apollonian circle and the pole, with regard to this circle, of the Brocard diameter are conjugate points with respect to the circumcircle.

15. The involution of conjugate points P_1, C_1; P_2, C_2; P_3, C_3 with respect to ω (14) is the section of the orthogonal involution of rays having its center at I; the rays joining I to the two points of intersection of 1 with any one of the Apollonian Circles are conjugate in this involution. Consequently:

The points of intersection of the Lemoine line with an Apollonian circle are conjugate with respect to the circumcircle.

16. Lemma. *If two tangents to two circles meet on the radical axis, the points of contact are collinear with one of the centers of similitude of the circles.*

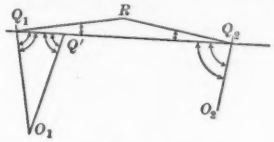

FIG. 2.

The points of contact Q_1, Q_2 and the point of intersection R of the tangents form an isosceles triangle, since $RQ_1 = RQ_2$. O_1 and O_2 being the centers of the circles, the angles $O_1Q_1Q_2$ and $O_2Q_2Q_1$ are equal, being the respective complements of two equal angles. Now, if Q' is the second point common to Q_1Q_2 and the circle O_1, we have $\angle O_1Q'Q_1 = \angle O_1Q_1Q_2 = \angle O_2Q_2Q_1$; hence O_1Q' is parallel to O_2Q_2, which proves the lemma.

This proof changes but slightly if O_1, O_2 are on opposite sides of Q_1Q_2.

17. The line $A'B'$ (Fig. 1) joining the points of contact A', B' of the two tangents OA', OB' to the two circles γ_1, γ_2, meets the line of their centers l in one of their centers of similitude (16). This is the point C_3 (12). The line AB' joining the points of contact A, B' of the two tangents OA, OB' to the two circles γ_1, γ_2 meets the line l in one of their centers of similitude (16). This is the point P_3 (13). Similarly for the points C_1, P_1, C_2, P_2. Hence:

The center of an Apollonian circle and the pole, with respect to this circle, of the Brocard diameter, are the two centers of similitude of the two other circles of Apollonius.

[1] Russell, *loc. cit.*, pp. 34, 35.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

A New Analysis of Plane Geometry, Finite and Differential. By A. W. H. THOMPSON, Sometime Scholar of Trinity College. Cambridge University Press, 1914. xvi + 120 pages.

From the preface of this interesting book we quote: "There are two main ways of considering Geometry. One is by sight or figure. The other is by a symbolic representation of the figure. The former method I call the Visual method; the latter the Symbolic method. . . . The method of the text belongs to the Symbolic method. In some cases, however, the process is easily visualized and coincides with the treatment of Visual Geometry. . . . I claim the method [of this text] as original. There are some theorems which are original, and most of the general results in the examples are new. The new treatment of the trigonometric functions is original and necessary for the purpose." The author remarks further upon the similarity of the symbols in this book and in the method of Grassman although he was not aware of Grassman's method when he discovered the method used here.

The book consists of two parts, Finite geometry and Differential geometry. The part on finite geometry contains an introduction and three chapters on the reduction of the three ways in which the elements of the geometry, points and lines, may be expressed. The nine chapters in the differential geometry are concerned with the differentiation of these expressions, the reduction of measures containing functional elements, generalized displacement of a point, the reduction of complex functional elements, the successive differentiation of measures, and integration of measures.

In the notation of this book the small Italic letters $a, b, c, \cdots, x, y, z$ denote points and the small Greek letters $\alpha, \beta, \gamma, \cdots, \omega$ denote lines. The determinate of two points is the line incident in the points and is denoted by $\overline{xy}$, while the determinate of two lines $\alpha\beta$ is their point of intersection. The measure of two points (xy) is the distance between them; the measure of two lines $(\alpha\beta)$ is the angle between them; the measure of a point and a line $(a\gamma)$ is the perpendicular distance from a to γ. Then $(\overline{ab}\ \overline{\gamma\delta})$ would be the perpendicular from the point of intersection of the two lines to a line determined by the points a and b. The measure (xy) is distinct from (yx) and the measure $(\alpha\beta)$ is distinct from $(\beta\alpha)$ so that lines and angles have a "sense" of direction. The general vectorial point may be expressed as $a_{\rho'\theta'\phi'}$ where ρ' is a vector having a direction ρ while the distance of the point from a is denoted by the magnitude ρ. The line $a_{\rho'\theta'\omega}$ is incident in the point $a_{\rho'\theta'}$ and has the direction ω. Furthermore lines and points may be expressed in a third way in equational elements as the point $A_r(\xi a_r) = 0$. The measure (xyz) is shown to be equal to twice the area of the triangle of which the points x, y, and z are the vertices. The chief part of the first three chapters is the reduction of measures of several elements to those of two elements. The definition of this method of reduction is not expressly given

but when one supplies this omission, the remainder of the process follows easily. Thus the definition $(xyz) = (xy)(\overline{xy}z)$ proves the theorem just mentioned. The definition is more apparent in $(\alpha\beta cd) = \sin(\alpha\beta)(\overline{\alpha\beta}cd) = \sin(\alpha\beta)(\overline{\alpha\beta}c)(\overline{\overline{\alpha\beta}}cd)$ which reduces to $(c\alpha)(d\beta) - (c\beta)(d\alpha)$. All the formulas derived for the first method of representation of points and lines are applicable to the vectorial and equational representations. The following problem given as an exercise may be proved by two very simple steps: Let $\left.\begin{matrix} abc \\ \alpha\beta\gamma \end{matrix}\right\}$ be a triangle and λ any line: p, q, r are the feet of the perpendiculars from a, b, c on λ: show that the perpendiculars from p, q, r on α, β, γ, are concurrent.

In the differential geometry, if x is any point, and x' a consecutive point, then (xx') is denoted by δx and $\overline{xx'}$ by τx, where τx is the line tangent to a curve at a point x upon it. νx is the normal line to a curve at a point x upon it. If ξ is any line and ξ' is a consecutive line, then $(\xi\xi')$ is denoted by $\partial\xi$ and $\overline{\xi\xi'}$ by $p\xi$. From these simple conditions are found the derivatives of the measures which were deduced in the first three chapters.

At the end of the book are given 108 formulas which are derived in the body of the work. At the end of each chapter is a collection of examples and following Chapter XII is a list of 106 miscellaneous examples. The method of the book may be shown best by a simple problem. "Show that the product of the perpendiculars from the foci of an ellipse to any tangent is constant."

Let $-|(xs)|+|(xs')| = k$ be the equation of the ellipse where s and s' are the foci and x is a variable point upon the curve.

Differentiating,

$$\frac{(xs)^2}{(\nu xs)^2} = \frac{(xs')^2}{(\nu xs')^2},$$

then

$$\frac{(\nu xs)}{(\tau xs)} + \frac{(\nu xs')}{(\tau xs')} = 0, \quad \text{that is,} \quad \frac{(\nu\xi s)}{(\xi s)} + \frac{(\nu\xi s')}{(\xi s')} = 0,$$

by putting $\tau x = \xi$.

Integrating,

$$\log(\xi s) + \log(\xi s') = \text{constant},$$

$$\therefore \quad (\xi s)(\xi s') = \text{constant}.$$

While this is entirely symbolic, the reader may obtain some idea of the beauty of this method by drawing the figure and seeking a visual interpretation of each step. Of course we have given no clue to the methods of differentiation or integration. We see in this book several theorems on the cross-ratio of a pair of points a, b and a pair of lines γ, δ, whose definition is $\dfrac{(a\gamma)/(b\gamma)}{(a\delta)/(b\delta)}$ and is written $\{ab, \gamma\delta\}$. Another interesting problem is, show that $(xa\beta c\delta ex) = 0$ is the equation of a conic and deduce Pascal's theorem.

This geometry will never replace the older geometries, but with further development many new theorems may be discovered. Any one interested in modern geometry will read the work with pleasure; and detecting the few slight errors will add a little zest.

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PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

PROBLEMS FOR SOLUTION.

ALGEBRA.

443 Proposed by A. M. KENYON, Purdue University.

If p_r denote the sum of all the different r-factor products that can be formed from the first n natural numbers ($p_r = 0$ for $r > n$), and if

$$D_s = \begin{vmatrix} p_1 & 1 & 0 & \cdots & 0 \\ 2p_2 & p_1 & 1 & \cdots & 0 \\ 3p_3 & p_2 & p_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ sp_s & p_{s-1} & p_{s-2} & \cdots & p_1 \end{vmatrix}$$

show that

$$\sum_{i=0}^{k} (-1)^i c_i \binom{k}{i} D_{2k-i} = 0, \qquad k, n = 1, 2, 3, \cdots,$$

where $c_i = \dfrac{2k+1-i}{1+i}$ when i is even and $2n+1$ when i is odd; and $\binom{k}{i}$ is the coefficient of x^i in $(1+x)^k$.

444. Proposed by J. E. ROWE, Pennsylvania State College.

Prove that the determinant

$$\begin{vmatrix} \cot A & \cot B & \cot C \\ 1 & 1 & 1 \\ \cos^2 A & \cos^2 B & \cos^2 C \end{vmatrix} = 0,$$

where A, B, and C are the angles of a plane triangle.

GEOMETRY.

474. Proposed by LAENAS G. WELD, Pullman, Illinois.

Upon a fixed and constant base stands a system of co-planar triangles, for each of which the radius of the inscribed circle is to that of the circumscribed circle as 1 : 2. What is the locus of the vertices opposite to the given fixed base?

475. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Given two circles and a straight line, to draw a circle tangent to the line and coaxial with the two given circles.

CALCULUS.

395. Proposed by W. W. BURTON, Mercer University, Macon, Ga.

Into a full conical wine glass whose depth is a and whose angle at the base is 2α there is carefully dropped a spherical ball of such size as to cause the greatest overflow. Show that the radius of the ball is $a \sin \alpha/(\sin \alpha + \cos 2\alpha)$.

From Woods and Bailey's *A Course in Mathematics* (1907), Volume I, page 213.

396. Proposed by ELBERT H. CLARKE, Purdue University.

The length of the curve $y = x^n$ from the origin to the point (1, 1) is given by the formula

$$l = \int_0^1 \sqrt{1 + n^2 x^{2n-2}}\, dx.$$

Our geometric intuition would tell us that the limit of this length as n becomes infinite is 2. Give a strict analytic proof that

$$\lim_{n \to \infty} \int_0^1 \sqrt{1 + n^2 x^{2n-2}}\, dx = 2.$$

MECHANICS.

315. Proposed by H. S. UHLER, Yale University.

A solid, homogeneous, right, circular cylinder is allowed to move from rest down a circular cylindrical track which is concave upwards. Find the ratio of the radius of the track to the radius of the cylinder when the time of descent through a finite arc to the bottom is the same for the extreme cases of no slipping and zero friction. Show also that the same relation holds for a sphere descending a cylindrical or spherical surface.

316. Proposed by C. N. SCHMALL, New York, N. Y.

A body at rest at a point R begins to move towards a center of force F. The distance $RF = d$, and the force varies inversely as the distance. Two intermediate points in the path are P and Q, such that $FP = kd$, and $FQ = k^n d$. Show that the body will traverse the distance QP in a *maximum* of time if $k = 1/n^{\frac{1}{2(n-1)}}$.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

426. Proposed by HERBERT N. CARLETON, West Newbury, Mass.

Find all solutions of the equation

$$x^{\sqrt{x}} = x^x.$$

SOLUTION BY J. A. CAPRON, Notre Dame, Ind.

The equation may be written in the form $x^{1+(1/x)} = x^x$, or $x^{(x+1)/x} - x^x = 0$. Factoring, we have $x^x[x^{(x+1-x^2)/x} - 1] = 0$. This equation is equivalent to the two equations $x^x = 0$ and $x^{(x+1-x^2)/x} - 1 = 0$. The first of these equations is satisfied for the value of $x = -\infty$. From the second equation, we have, by taking logarithms, the equation

$$\left(\frac{x + 1 - x^2}{x}\right) \log x = 0.$$

This equation is equivalent to the three equations $1/x = 0$, $x + 1 - x^2 = 0$, and $\log x = 0$. From the first of these equations, $x = \pm \infty$; from the second, $x = (1 \pm \sqrt{5})/2$; and from the third, $x = 1$.

By substituting the values of x found above in the original equation, we see that 0 and $\pm\infty$ are to be rejected. We find, however, by inspection that $x = 1$ is a root.

Hence, the roots are 1 and $(1 \pm \sqrt{5})/2$.

Also solved by ALBERT N. NAUER, A. M. HARDING, C. E. GITHENS, V. M. SPUNAR, ELIJAH SWIFT, W. C. EELLS, G. W. HARTWELL, and the PROPOSER.

430A. Proposed by H. C. FEEMSTER, York College, Neb.

Solve the equations

$$\sum_{i=1}^{n} x_i - x_n = k + \frac{n^2 - 3n + 2}{2} d, \tag{1}$$

$$\sum_{i=1}^{n} x_i - x_{n-1} = k + \frac{n^2 - 3n + 4}{2} d, \tag{2}$$

$$\sum_{i=1}^{n} x_i - x_{n-2} = k + \frac{n^2 - 3n + 6}{2} d, \tag{3}$$

$$\vdots \qquad \vdots$$

$$\sum_{i=1}^{n} x_i - x_1 = k + \frac{n^2 - n}{2} d. \tag{n}$$

SOLUTION BY A. M. HARDING, Univ. of Arkansas.

Add the given equations and obtain

$$(n-1)\sum_{i=1}^{n} x_i = nk + \frac{n^3 - 3n^2 + n^2 + n}{2} d,$$

or

$$\sum_{i=1}^{n} x_i = \frac{n}{n-1} \cdot k + \frac{n(n-1)}{2} d.$$

Subtract each of the given equations from this equation and obtain

$$x_n = \frac{k}{n-1} + (n-1)d,$$

$$x_{n-1} = \frac{k}{n-1} + (n-2)d,$$

$$x_{n-2} = \frac{k}{n-1} + (n-3)d,$$

$$\vdots$$

$$x_2 = \frac{k}{n-1} + d,$$

$$x_1 = \frac{k}{n-1}.$$

Also solved by NATHAN ALTSHILLER, S. A. JOFFE, J. W. CLAWSON, FRANK R. MORRIS, ELBERT H. CLARKE, HORACE OLSON, N. P. PANDYA, and the PROPOSER.

431. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Form a magic square of 9 cells such that (the integers being all different) the products of the integers in the rows, columns, and diagonals shall be the same and the smallest product possible.

SOLUTION BY S. A. JOFFE, New York City.

All the nine integers will be determined if, in addition to the product p of three integers of any row (column, or diagonal), we know two extremes, say of the first row, a and b, and the central integer c.

a		b
	c	

The extremes of the third row will then be p/ac and p/bc; of the second row will be bc/a and ac/b; and those of the second column will be p/ab and ab/c.

Hence the product of the integers in the second row equals

$$\frac{bc}{a} \cdot c \cdot \frac{ac}{b} = c^3,$$

and therefore

$$p = c^3,$$

so that the magic square becomes

a	$\frac{c^2}{ab}$	b
$\frac{bc}{a}$	c	$\frac{ac}{b}$
$\frac{c^2}{b}$	$\frac{ab}{c}$	$\frac{c^2}{a}$

If we disregard the central integer c, the remaining eight integers must therefore form four pairs: (1) m_1 and c^2/m_1; (2) m_2 and c^2/m_2; (3) m_3 and c^2/m_3; (4) m_4 and c^2/m_4. Taking now for m_1, m_2, m_3 and m_4 the smallest four integers, *i. e., 1, 2, 3 and 4,* we find that the *smallest square* number c^2, divisible by these four integers, is *36,* and consequently $c = \sqrt{36} = 6$.

However, no corner number, say a, can be unity; because, if $a = 1$ then ac/b and ab/c become c/b and b/c, which must be simultaneously integers, and this is impossible unless $b = c$; but the latter equality is excluded by the condition of the problem. Taking $a = 2$, $b = 3$ and $c = 6$, we find the required magic square with the smallest product possible to be

2	36	3
9	6	4
12	1	18

Also solved by HERBERT N. CARLETON, N. P. PANDYA.

432. Proposed by C. N. SCHMALL, New York City.

There are n straight lines in a plane, no two of which are parallel and no three of which are concurrent. Their points of intersection being joined show that the number of new lines drawn is $\frac{1}{8}n(n-1)(n-2)(n-3)$.

SOLUTION BY LAENAS G. WELD, Pullman, Illinois.

Each of the n given lines intersects each of the other $(n-1)$ lines, determining (since each point is thus twice determined) points to the number of $\frac{n(n-1)}{2}$.

Let P_{ik} be the point determined by the lines l_i and l_k. From P_{ik} there may be drawn $R\left(=\frac{n(n-1)}{2}-1\right)$ lines to the other points. But the line l_i contains $(n-2)$ of these other points and the line l_k the same number. Hence the given lines l_i and l_k account for $2(n-2)$ of the above R lines and the number of new lines is

$$R-2(n-2)=\left(\frac{n(n-1)}{2}-1\right)-2(n-2).$$

Since there are $\frac{n(n-1)}{2}$ points such as P_{ik} the total number of new lines (since each is thus counted twice) is

$$\frac{1}{2}\cdot\frac{n(n-1)}{2}\left\{\left(\frac{n(n-1)}{2}-1\right)-2(n-2)\right\},$$

or

$$\frac{1}{8}n(n-1)(n-2)(n-3).$$

Also solved by J. W. CLAWSON, HERBERT N. CARLETON, H. C. FEEMSTER, WALTER C. EELLS, PAUL CAPRON, FRANK R. MORRIS, ELBERT H. CLARKE, N. P. PANDYA, and HORACE OLSON.

GEOMETRY.

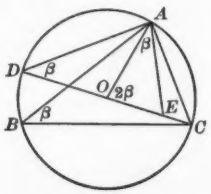
461. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Prove by means of any inscribed triangle the following trigonometrical relations:

$$\sin 2\beta = 2\sin\beta\cos\beta; \qquad \cos 2\beta = \cos^2\beta - \sin^2\beta; \qquad \sin 3\beta = 3\sin\beta - 4\sin^3\beta;$$
$$\cos 3\beta = 4\cos^3\beta - 3\cos\beta.$$

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

Let ABC be the triangle. Join A and C to O, the center of the circumcircle. Extend CO to meet the circle at D. Draw AD. Draw AE making angle OAE equal to angle ABC. Denote angle ABC by β. Call the radius R.



$$(1)\qquad \sin 2\beta = \sin AOC = \sin ACD \cdot \frac{AC}{R} = \frac{AC \cdot AD}{2R^2} = 2\frac{AC}{2R} \cdot \frac{AD}{2R} = 2 \sin\beta \cos\beta.$$

$$(2)\qquad \cos 2\beta = \cos AOC = \frac{R^2 + R^2 - \overline{AC}^2}{2R \cdot R} = \frac{4R^2 - \overline{AC}^2 - \overline{AC}^2}{4R^2} = \frac{\overline{AD}^2 - \overline{AC}^2}{4R^2}$$

$$= \left(\frac{AD}{2R}\right)^2 - \left(\frac{AC}{2R}\right)^2 = \cos^2\beta - \sin^2\beta.$$

$$(3)\qquad \sin 3\beta \sin AEC = \sin AEO = \sin\beta \cdot \frac{R}{OE}.$$

Now

$$\frac{OE}{R} = \frac{AE}{AD} = \frac{AE}{2R\cos\beta}; \qquad i.\,e.,\ AE = 2\overline{OE}\cos\beta.$$

Again

$$OE^2 = \overline{AE}^2 + R^2 - 2\overline{AE} \cdot R\cos\beta = 4\overline{OE}^2\cos^2\beta + R^2 - 4\overline{OE} \cdot R \cdot \cos^2\beta;$$

$$\therefore\ OE = \frac{R}{4\cos^2\beta - 1};$$

$$\therefore\ \sin 3\beta = \sin\beta(4\cos^2\beta - 1) = 3\sin\beta - 4\sin^3\beta.$$

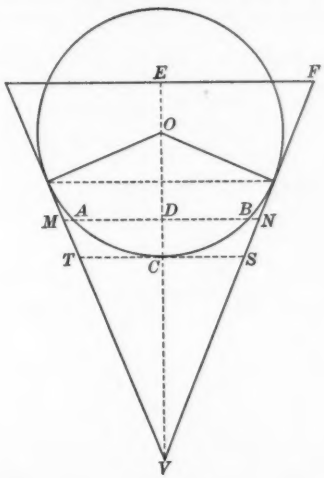
$$(4)\qquad \cos 3\beta = \frac{R^2 - \overline{AE}^2 - \overline{OE}^2}{2\overline{OE} \cdot AE} = \frac{(4\cos^2\beta - 1)^2 - 1 - 4\cos^2\beta}{4\cos\beta} = 4\cos^3\beta - 3\cos\beta.$$

462. Proposed by DANIEL KRETH, Wellman, Iowa.

A conical glass, the diameter of the base of which is 5 inches and altitude 6 inches, is one-fifth full of water. If a sphere 4 inches in diameter is dropped into it, how much of the vertical axis of the glass is immersed?

SOLUTION BY J. A. CAPRON, Notre Dame University.

Let MN be the level of the water after the sphere is dropped; $R = 5/2$ inches = radius of base of given cone; $r = 2$ inches = radius of sphere; r_1 = radius of base of cone VMN; x = height of spherical segment ACB; α = semivertical angle of cone; and $CV = a$.



Then volume of cone $VMN = V_c = \frac{1}{3}\pi r_1^2(a+x)$; volume of segment $ACB = V_s = \pi x^2(r - x/3)$; and volume of given cone $= V = \frac{1}{3}\pi R^2 h$, where $h = 6$ inches.

Hence $V_e - V_s = \frac{1}{5}V$, since $V_e - V_s$ = volume of water. Substituting, we get

$$r_1^2(a + x) - x^2(3r - x) = \frac{R^2h}{5}.$$

Since

$$VC = VO - CO \qquad \text{and} \qquad VO = \frac{r}{\sin \alpha},$$

we get

$$a = r\left(\frac{1}{\sin \alpha} - 1\right),$$

but since

$$\tan \alpha = \frac{5}{12}, \qquad \sin \alpha = \frac{5}{13},$$

therefore

$$a = 2\left(\frac{13}{5} - 1\right) = \frac{16}{5}.$$

From triangles VEF and VDN, we get

$$r_1 = \frac{(a + x)}{h}R, \qquad \text{or} \qquad r_1 = \frac{\left(\frac{16}{5} + x\right)\frac{5}{2}}{6} = \frac{4}{3} + \frac{5x}{12}.$$

Substituting these values in the above equation and reducing, we get

$$845x^3 - 3120x^2 + 3840x - 1304 = 0.$$

To solve this equation, let

$$x = y + \frac{108}{169}, \qquad \text{or} \qquad x = y + \frac{16}{13}.$$

Substituting and reducing, we get

$$169y^3 = -\frac{16^3}{13} + 260.8, \qquad \text{or} \qquad 13^3y^3 = -705.6.$$

Hence,

$$y = -\frac{\sqrt[3]{705.6}}{13} = -\frac{2}{13}\sqrt[3]{88.2}.$$

Then

$$x = y + \frac{16}{13} = \frac{16 - 2\sqrt[3]{88.2}}{13} = .54595.$$

Hence, the required height $= a + x = 3.2 + .54595 = 3.74595$ inches.

Excellent solutions were received from NATHAN ALTSHILLER, C. N. SCHMALL, HERBERT N. CARLETON, J. W. CLAWSON, HORACE OLSON, and PAUL CAPRON.

CALCULUS.

375. Proposed by V. M. SPUNAR, Chicago, Illinois.

Solve the differential equation

$$x^2(a - bx)\frac{d^2y}{dx^2} - 2x(2a - bx)\frac{dy}{dx} + 2(3a - bx)y = 6a^2.$$

I. SOLUTION BY H. T. BIGELOW, La Fayette, Indiana.

Let

$$y = \sum_{n=0}^{\infty} c_n x^n, \qquad \frac{dy}{dx} = \sum_{n=0}^{\infty} n c_n x^n, \qquad \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}.$$

Substituting and collecting terms we have,

$$\sum_{n=0}^{\infty} a(n^2 - 5n + 6)c_n x^n \equiv \sum_{n=0}^{\infty} b(n^2 - 3n + 2)c_n x^{n+1} + 6a^2.$$

Equating the coefficients of each power of x on the two sides of this equality, we have,

$$\text{for } x^0 \qquad 6ac_0 = 6a^2;$$

$$\text{for } x \qquad 2ac_1 = 2bc_0;$$

$$\text{for } x^2 \qquad 0ac_2 = 0bc_1;$$

$$\text{for } x^3 \qquad 0ac_3 = 0bc_2;$$

and

$$\text{for } x^i, \ (i^2 - 5i + 6)aci = (i^2 - 5i + 6)bc_{i-1}.$$

Solving these equations;

$$c_0 = a, \qquad c_1 = b, \qquad c_2 \text{ is arbitrary}, \qquad c_3 \text{ is arbitrary},$$

$$c_4 = \frac{b}{a}c_3, \qquad c_5 = \frac{b^2}{a^2}c_3, \qquad c_6 = \frac{b^3}{a^3}c_3, \quad \text{etc.}$$

Hence,

$$y = a + bx + c_2x^2 + c_3\left(x^3 + \frac{b}{a}x^4 + \frac{b^2}{a^2}x^5 + \cdots\right)$$

$$= a + bx + c_2x^2 + \frac{c_3x^3}{1 - \frac{b}{a}x}, \qquad \text{if } |x| < \frac{a}{b}.$$

Renaming the constants,

$$y = a + bx + Ax^2 + \frac{Bx^3}{a - bx},$$

where A and B are arbitrary. By substitution, it is readily verified that this actually is the solution for all values of x, except of course $x = \frac{a}{b}$.

II. Solution by W. W. Beman, Ann Arbor, Mich.

This problem appears in Article 69 of Forsyth's *Differential Equations* and may be solved by the method there indicated. The resolution into factors may be effected more easily by symbolic methods.

Putting

$$x\frac{d}{dx} \equiv \theta,$$

the equation becomes

$$[a(\theta^2 - 5\theta + 6) - bx(\theta^2 - 3\theta + 2)]y = 6a^2,$$

which may be written in three different ways:

$$[a(\theta - 3) - bx(\theta - 1)](\theta - 2)y = 6a^2, \text{ or}$$

$$(\theta - 3)[a(\theta - 2) - bx(\theta - 1)]y = 6a^2, \text{ or}$$

$$(\theta - 2)[a(\theta - 3) - bx(\theta - 2)]y = 6a^2.$$

Hence, three particular integrals of the equation, with second member 0, are

$$y_1 = x^2, \qquad y_2 = \frac{x^2}{a - bx}, \qquad y_3 = \frac{x^3}{a - bx},$$

only two of which are independent.

We may now put $y = y_1 v$, or $y = y_2 w$, or $y = y_3 z$, etc.: or $(\theta - 2)y = v$, or[1] $[a(\theta - 2) - bx(\theta - 1)]y = w$, or $[a(\theta - 3) - bx(\theta - 2)]y = z$, etc.

The substitution $y = y_2 w$ leads to the *normal* form, which could have been obtained in the ordinary way,

$$\frac{d^2 w}{dx^2} = \frac{6a^2}{x^4}.$$

Hence,

$$w = A + Bx + \frac{a^2}{x^2}.$$

$$\therefore \quad y = \frac{x^2}{a - bx}\left[A + Bx + \frac{a^2}{x^2}\right].$$

Again, the equation may be written

$$[(\theta - 2)(\theta - 3)(a - bx)]y = 6a^2.$$

$$\therefore \quad (a - bx)y = Ax^2 + Bx^3 + a^2.$$

Also solved by V. M. SPUNAR, ELIJAH SWIFT, A. M. HARDING, and ELMER SCHUYLER.

376. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that

$$\frac{1}{z} - \frac{1}{z}(1 - 2xz + z^2)^{1/2} = x + \frac{z}{2}\left(\frac{x^2 - 1}{1 - xz}\right) + \sum_{n=2}^{n=\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 3)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}(x^2 - 1)^n \left(\frac{z}{1 - xz}\right)^{2n-1}$$

SOLUTION BY ELIJAH SWIFT, University of Vermont.

First of all the coefficients of the terms of the infinite series are the same as those in the expansion of $-(1 - a)^{\frac{1}{2}}$. These lead us to expand

$$\left\{1 - (x^2 - 1)\left(\frac{z}{1 - xz}\right)^2\right\}^{1/2}$$

by the Binominal theorem, which gives

$$1 - \tfrac{1}{2}(x^2 - 1)\left(\frac{z}{1 - xz}\right)^2 - \left\{\frac{1 \cdot 1}{2 \cdot 4}(x^2 - 1)^2 \left(\frac{z}{1 - xz}\right)^4 + \cdots\right\}.$$

Noting that the terms in the bracket are the corresponding terms in the infinite series each multiplied by $z/(1 - xz)$, we find that

$$\sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 3)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}(x^2 - 1^n)\left(\frac{z}{1 - xz}\right)^{2n-1}$$
$$= -\frac{1 - xz}{z}\left\{1 - (x^2 - 1)\left(\frac{z}{1 - xz}\right)^2\right\}^{1/2} + \frac{1 - xz}{z} - \tfrac{1}{2}(x^2 - 1)\left(\frac{z}{1 - xz}\right).$$

[1] These two forms of solution are indicated in the German Edition of Forsyth.

Substituting this value, the result follows by a simple algebraic reduction. The result is true only when the series converges, *i. e.*, when $|(x^2-1)(z/(1-xz))^2|<1$ or $x < (1+z^2)/2z$.

Also solved by S. A. JOFFE, GEO. W. HARTWELL, ALBERT N. NAUER, and A. M. HARDING. *Note.*—This problem should have been placed in the Algebra department. EDITORS.

377. Proposed by W. D. CAIRNS, Oberlin College.

It is required to find a curve of the form $y = x(x-a)(x-b)$ such that the abscissas of the maximum and minimum values, as well as a and b, shall be positive integers.

SOLUTION BY RALPH D. BEETLE, Dartmouth College.

The abscissas of the minimum and maximum values are

$$(1) \qquad x_1 = \tfrac{1}{3}[a+b+\sqrt{a^2-ab+b^2}], \qquad x_2 = \tfrac{1}{3}[a+b-\sqrt{a^2-ab+b^2}].$$

If a and b are positive integers, so also are x_1 and x_2 if, and only if, $a+b$ is divisible by 3 and a^2-ab+b^2 is a perfect square divisible by 9. These conditions are equivalent to the simpler ones that a and b be divisible by 3 and a^2-ab+b^2 be a perfect square. If we write

$$(2) \qquad a = 3h, \; b = 3k$$

our problem is reduced to the finding of positive integral values of h and k such that h^2-hk+k^2 is a perfect square. We note at once that, if a and b are equal, it is sufficient that they be multiples of 3. We may find all solutions of the problem as follows:

Without loss of generality, we may assume that $h \leqq k$, so that $h^2-hk+k^2 \leqq k^2$. If we put $h^2-hk+k^2=(k-r)^2$, it is found that

$$(3) \qquad h = \tfrac{1}{2}[k \pm \sqrt{k^2-8rk+4r^2}].$$

In order that h be an integer, we must have $k^2-8rk+4r^2=t^2$, where t is an integer, or $(k-4r)^2-t^2=12r^2$. It is evident that k and t are both even or both odd, and we may therefore put

$$(4) \qquad k-4r = n+s, \qquad t = n-s,$$

where n and s are positive integers. Then

$$(5) \qquad ns = 3r^2,$$

and the values of a, b, x_1, x_2 in terms of r, n, s are found from (1), (2), (3), (4) to be either

$$a = 3n+6r \qquad b = 3n+12r+3s, \qquad x_1 = 3n+9r+2s, \qquad x_2 = n+3r,$$

or

$$a = 6r+3s, \qquad b = 3n+12r+3s, \qquad x_1 = 2n+9r+3s, \qquad x_2 = 3r+s.$$

Since the two sets differ only in the interchange of n and s, only one set is needed in view of the symmetric roles of n and s in the relation (5). We may therefore state our result as follows:

If n, s and r are integers such that $n > 0$, $s \geqq 0$, $r \geqq 0$, $ns = 3r^2$, and if $a = 3n + 6r$, $b = 3n + 12r + 3s$, the abscissas of the minimum and maximum points of the curve, $y = x(x - a)(x - b)$, are the positive integers $x_1 = 3n + 9r + 2s$ and $x_2 = n + 3r$.

The smallest unequal values of a and b are 9 and 24. In the following table are the solutions which correspond to $r = 1$ and $r = 2$.

n	1	3	1	2	3	4	6	12
s	3	1	12	6	4	3	2	1
r	1	1	2	2	2	2	2	2
a	9	15	15	18	21	24	30	48
b	24	24	63	48	45	45	48	63
x_1	18	20	45	36	35	36	40	56
x_2	4	6	7	8	9	10	12	18

The consideration of negative values of n and s is unnecessary, since it can be proved that no additional solutions of the problem are thus obtained.

Also solved by ALBERT N. NAUER, W. C. EELLS, JOSEPH B. REYNOLDS, ELMER SCHUYLER ELIJAH SWIFT, HORACE OLSON, and the PROPOSER.

MECHANICS.

292. Proposed by C. N. SCHMALL, New York City.

In a bombardment, a battleship directs its fire at a fort standing on a hill whose height is a feet above the sea level. The angle of elevation of the fort is found to be ϕ. If the initial velocity of the projectile is v, show that the fort will *not* be struck if $v < \sqrt{ag(1 + \csc \phi)}$.

SOLUTION BY H. S. UHLER, Yale University.

Let the coördinate plane contain the fort and the vertical through the ship. Also let θ denote the angle which the rifle makes with the positive (upward) direction of the axis of y. At any point of the trajectory

$$x = v \sin \theta \cdot t,$$

$$y = v \cos \theta \cdot t - \tfrac{1}{2}gt^2,$$

hence

$$gx^2 - v^2 \sin 2\theta \cdot x + 2v^2 \sin^2 \theta \cdot y = 0.$$

This parabola will pass through the fort ($x = a \cot \phi$, $y = a$) when

$$a^2 g \cot^2 \phi \cdot \cot^2 \theta - 2av^2 \cot \phi \cdot \cot \theta + a(2v^2 + ag \cot^2 \phi) = 0.$$

The roots of this quadratic in $\cot \theta$ will be complex if

$$(v^2 - ag)^2 - (ag \csc \phi)^2$$

is negative. Since this expression may be written

$$[v^2 + ag(\csc \phi - 1)] \cdot [v^2 - ag(\csc \phi + 1)],$$

it is seen at once that the fort will not be hit when

$$v < \sqrt{ag(1 + \csc \phi)}.$$

295. Proposed by B. F. FINKEL, Drury College.

A homogeneous hollow cylinder whose inner radius is half its outer radius, rolls without slipping down a plane inclined at an angle α with the horizontal. Find its acceleration.

SOLUTION BY JOS. B. REYNOLDS, Lehigh University.

If in a section perpendicular to the axis of the cylinder, O is the center and A the point of contact with the plane, we shall have for the moment of inertia about O,

$$I_O = \frac{\pi}{2} e \left(r^4 - \frac{r^4}{2^4} \right),$$

and for the mass

$$m = \frac{W}{g} = \pi e \left(r^2 - \frac{r^2}{2^2} \right),$$

whence

$$I_O = \frac{5}{8} \frac{W}{g} r^2$$

and therefore

$$I_A = I_O + \frac{W}{g} r^2 = \frac{13}{8} \frac{W}{g} r^2.$$

Now taking moments about A, we have $Wr \sin \alpha = I_A \ddot{\theta}$ where $\ddot{\theta}$ is the angular acceleration, or

$$Wr \sin \alpha = \frac{13}{8} \frac{W}{g} r^2 \ddot{\theta},$$

whence $r\ddot{\theta} = \frac{8}{13} g \sin \alpha$, the linear acceleration of the center of the cylinder.

Also solved by H. S. UHLER.

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas.

REPLIES.

24. The following facts are significant:

(1) The New England Association of Mathematics Teachers has appointed a committee "to investigate the current criticisms of high school mathematics."

(2) A committee of the Council of the American Mathematical Society has under consideration the question "whether any action is desirable on the part of the Society in the matter of the movement against mathematics in the schools."

(3) At the recent meeting in Cincinnati of the National Education Association an iconoclastic discussion on the topic: "Can algebra and geometry be reorganized so as to justify their retention for high school pupils not likely to enter technical schools?" aroused approbation and applause. An outline of the remarks by one of the speakers was printed in a previous issue.

In view of these facts what should be done by those who believe in the value of mathematics as a general high school study?

REPLY BY HARRISON E. WEBB, Central High School, Newark, N. J.

High-school teachers of mathematics should (1) do their utmost to restore the older ideal of high-school education as a serious business for the individual student; (2) relate their science as closely as possible to other studies which are pursued simultaneously; (3) preserve as closely as possible the essential continuity of the entire high-school mathematical course; (4) if necessary, as it may be, to accomplish these ends, eliminate excessive complications and omit less important topics; (5) give over entirely the notion that their subjects are absolutely logical, but make them positive so far as they go; and (6) believe in mathematics as possessing an incomparable moral value.

DISCUSSIONS.

RELATING TO INTERPOLATION IN TABLES OF TWO ARGUMENTS.

BY IRWIN ROMAN, Chicago, Ill.

Anyone who works with tables of two arguments has probably found interpolation inconvenient. The following method renders it possible with almost the same facility as interpolation in one argument. Let the function be given for four adjacent values, two in each argument. This may be represented schematically as follows. Let $A = f(x_1, y_1)$, $B = f(x_2, y_1)$, $C = f(x_1, y_2)$, $D = f(x_2, y_2)$ and let it be required to find $G = f(x_1 + \Delta x, y_1 + \Delta y)$ where

$$\frac{\Delta x}{x_2 - x_1} = e_x < 1 \quad \text{and} \quad \frac{\Delta y}{y_2 - y_1} = e_y < 1.$$

For convenience, let

$$x_1 \leqq x_1 + \Delta x \leqq x_2 \quad \text{and} \quad y_1 \leqq y_1 + \Delta y \leqq y_2.$$

Let

$$E = f(x_1, y_1 + \Delta y) \quad \text{and} \quad F = f(x_2, y_1 + \Delta y).$$

The usual way of finding G is to interpolate E from A and C, to interpolate F from B and D and finally, to interpolate G from E and F. This gives

$$E = A + e_y(C - A),$$

$$F = B + e_y(D - B),$$

$$G = E + e_x(F - E).$$

Carrying this step farther gives

$$G = A + e_y(C - A) + e_x[B + e_y(D - B) - A - e_y(C - A)]$$

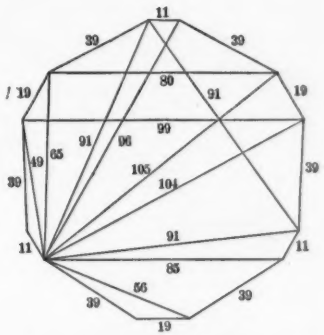
$$= A + e_y(C - A) + e_x(B - A) + e_x e_y[(A + D) - (B + C)].$$

The last term is usually negligible, especially if $f(x, y)$ changes in the same sense for both x and y. This is simple to remember as it involves an interpolation horizontally and one vertically. It avoids the necessity of calculating E and F.

RELATING TO A GEOMETRIC REPRESENTATION OF INTEGRAL SOLUTIONS OF CERTAIN QUADRATIC EQUATIONS.

By Norman Anning, Chilliwack, British Columbia.

The cyclic dodecagon shown in the accompanying figure is an interesting special solution of the problem: To locate n points in the plane so that the $\binom{n}{2}$ distances shall be integral.



The fact that Ptolemy's Theorem may be verified in no less than 81 different ways entitles the group of numbers to rank with the most dignified of magic squares. The 13 numbers are all found among the integral solutions of

$$x^2 \pm xy + y^2 = 7^2 \cdot 13^2.$$

In like manner 40 integers which occur among the solutions of

$$x^2 \pm xy + y^2 = 7^2 \cdot 13^2 \cdot 19^2$$

may be exhibited as the sides and diagonals of a cyclic 24-gon. The sides in order are: 96, 361, 299, 209, 249, 209, 299, 361, 96, 361, ...

That a study of

$$x^2 + xy + y^2 = 7^2 \cdot 13^2 \cdot 19^2 \cdot 31^2$$

would yield a similar 48-gon is ~~probable~~ a fact. A. A., Jan. 1945.

NOTES AND NEWS.

EDITED BY W. D. CAIRNS, Oberlin, Ohio.

Dr. W. S. FRANKLIN has resigned from his position as professor of physics in Lehigh University.

A "School algebra—first course," by RIETZ, CRATHORNE and TAYLOR appears in the press of Henry Holt and Company.

Dr. G. H. GRAVES, of Columbia University, has been appointed instructor in mathematics in Purdue University.

President ROBERT S. WOODWARD, of Carnegie Institution, and Professor ARTHUR G. WEBSTER, of Clark University, have been chosen from the American Mathematical Society to membership on the Naval Advisory Board of Inventions.

JOHN HOWARD VAN AMRINGE, professor emeritus of mathematics in Columbia University, a member of the Columbia faculty for fifty years up to his retirement in 1910, died September 10.

Mr. F. E. CARR has returned to a position in the department of mathematics in Oberlin College after two years of study in the graduate school of the University of Chicago.

In *Science* for September 10 Professor G. A. MILLER discusses critically the article by Professor E. Borel in the April number of *Revue du Mois*, in which he asserts that in this country, as in Germany, the prevailing tendency in estimating scientific achievements is to emphasize quantity rather than quality.

In the May–July number of *Rendiconti del Circolo Matematico di Palermo* appear two articles by Americans: "The restricted problem of three bodies," by Professor G. D. BIRKHOFF, and "On the degree of approximation to discontinuous functions by trigonometric sums," by CHARLES E. WILDER, of Cambridge, Mass.

The Macmillan Company has published "Modern instruments and methods of calculation," a hand-book of the Napier tercentenary exhibition, under the editorship of E. M. Horsburgh. The same press announces an "Historical introduction to mathematical literature" by Professor G. A. MILLER.

An address entitled "Forty years' fluctuations in mathematical research," by Professor H. S. WHITE, which was read before the Vassar faculty club and the Columbia University mathematical colloquium, is printed in *Science* for July 23.

N. C. GRIMES, formerly professor of mathematics in the University of Arizona and lately assistant in mathematics in the University of Illinois, has accepted a

position in the University of Oregon as professor of mathematics and secretary to the President.

Dr. NATHAN ALTSHILLER, of the mathematical staff in the University of Washington, has resigned there to accept an instructorship in mathematics in the University of Colorado, taking effect in September of the present year.

A meeting of Missouri teachers of collegiate mathematics was also held at St. Louis on Saturday, November 27, in conjunction with the Southwestern Section of the American Mathematical Society. A further report of this meeting will appear in the December MONTHLY.

Professor F. CAJORI has returned to his teaching in Colorado College after a year abroad. After attending the Napier tercentenary at Edinburgh, he spent most of the year in Oxford and London in research connected with the history of mathematics. Other parts of the year he spent in travel in France, Italy, Switzerland, and Germany.

The Fall Announcement of New Macmillan Books includes the title "Historical Introduction to Mathematical Literature," by Professor G. A. Miller. The work "aims to give a brief account of the most important modern mathematical activities such as the mathematical societies, mathematical congresses, and periodical publications."

J. K. SINCLAIR, professor emeritus of mathematics in Worcester Polytechnic Institute, died September 12 after two years of increasingly failing health. Because of his sickness the leave of absence of his daughter Dr. M. E. Sinclair of Oberlin College was extended for a year, her place being supplied by Mr. L. M. Coffin, a graduate of the University of Maine and a master of arts in the University of Michigan.

The North-Eastern Ohio Teachers' Association was held in Cleveland October 22 and 23. At the departmental meeting for mathematics Mr. P. C. Bickel of the Alliance high school presented a paper on "How much time per week for supervision in the study of mathematics?" and Mr. H. A. Tuttle of Rayen high school, Youngstown, a paper on "The present trend in mathematical teaching—is it wise?"

In *School Science and Mathematics* for October J. H. Weaver, of the West Chester (Pa.) high school, presents the principal methods of the Greeks for attacking "The trisection problem"; M. O. Tripp, of Olivet College, points out the importance of "Symmetry in elementary geometry"; and J. V. Collins, of the Stevens Point (Wis.) State Normal School, discusses "Rational *vs.* mechanical methods in teaching mathematics."

In the *Proceedings* of the Society for the Promotion of Engineering Education appears the preliminary report of the committee on coöperation with secondary schools, under the chairmanship of H. E. WEBB, Central Commercial and Manual Training High School, Newark, N. J. The report urges the fuller recognition of the worth of high-school training for its own sake, a greater willingness to accept at its face value the judgment of a high-school teacher as to the ability of his pupils, and the elimination of the waste between the lower and higher institutions which is at present involved in the repetition of numerous topics in mechanical drawing, manual training, mathematics, physics and chemistry.

The sixth number of the first volume of the *Proceedings of the National Academy of Sciences* contains an article by Professor F. R. MOULTON on the "Solution of an infinite system of differential equations of the analytic type" and another by Professor HENRY BLUMBERG entitled "On the factorization of various types of expressions." The seventh number contains articles by Professor W. B. FORD on "The representation of arbitrary functions by definite integrals" and by Professor W. D. MACMILLAN on "Some theorems connected with irrational numbers." The eighth number contains an article by Professor H. S. WHITE on "Seven points on a twisted cubic curve."

The first meeting of the Kansas association of teachers of college mathematics was held at Topeka, Kansas, November 12. This meeting was the result of a movement initiated in the spring of 1915 for the improvement of teaching collegiate mathematics in the colleges of Kansas. It is a part of a nation wide movement having the same end. The temporary committee of organization consisted of Professor A. J. HOARE of Fairmount College, Professor T. J. MERGENDAHL of Emporia College, and Professor S. LEFSCHETZ of the University of Kansas. About fifteen representatives attended the meeting and a permanent organization was effected with Professor HOARE as president. A paper was presented by Professor W. A. HARSHBARGER of Washburn College on "What courses for the mathematical student beyond the calculus." The paper was discussed by Professor B. L. REMICK of the State Agricultural College and Professor W. H. GARRET of Baker University. It will appear in the December issue of the MONTHLY as a reply to question 27 in the department of Questions and Discussions.

This action of the college teachers of mathematics in Kansas is the first step in a movement that promises to grow rapidly. Definite plans are already formed for a similar organization in Ohio during the Christmas holidays, in conjunction with the meeting to be called for organizing a new national mathematical association, which is to be held at Columbus on Thursday, December 30, at ten o'clock in Page Hall of Ohio State University.

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AN EXPERIMENT IN CORRELATING FRESHMAN MATHEMATICS.

By F. L. GRIFFIN, Reed College.

Perhaps everyone would agree that it is desirable (if it is feasible) to introduce the study of calculus much earlier than is customary, to correlate it more closely with the elementary subjects and these in turn more closely with each other, and to exhibit actual uses for each topic at the time it is taught.

Quite commonly, unless a student presents trigonometry and college algebra for admission to college, he does not reach integral calculus until the middle of his sophomore year, and in some excellent colleges not until his junior year. Consequently not even the simplest integrations can be employed in the early physics courses. Moreover, students of the natural or social sciences who need a little knowledge of integral calculus must take two or three years of college mathematics in order to get it. Those who can devote but one year to mathematics in college must usually stop with no idea at all of mathematics beyond trigonometry and college algebra, and a very inadequate conception of the uses of even these subjects.

In fact, how *can* the matter of applications be adequately treated when the different branches are studied separately? Few practical problems depend for their solution upon a single branch of mathematics. Such topics as trigonometric analysis show their full value only by the closest sort of correlation.

Again, when we see 200 freshmen spend a couple of weeks decomposing the most complicated fractions into partial fractions, in order that 20 of their number may be able to use this process two years later in the integral calculus, we not only question the value of the process for the freshmen but we are inclined to doubt the law of the conservation of energy. This is an extreme case; but, wherever a student must go through college algebra, trigonometry and analytic geometry before getting any calculus, similar difficulties exist.

Of course, several institutions have partially overcome these difficulties for students of the sciences and others who wish to get a glimpse of higher mathematics, by offering survey courses designed primarily for those who do not expect to specialize in mathematics. But how about the students who *do* intend to specialize—who are expecting to work in engineering, astronomy, or physics,